Prova del 23/01/2012

1. $$\lim_{x \to 2} \log_2 (x+2) = 2$$

2. $$f(x) = \frac{2}{x\sqrt{x+4}}$$

3. $$f(x) = \begin{cases} \ln(e+x) + \alpha & x > 0 \\ 1 + \beta \tan x & x \leq 0 \end{cases}$$

4. $$A = \left\{ \frac{3^n}{1+3^n} : n \in \mathbb{N} \right\} \leftarrow$$

5. $$\lim_{x \to 0} \frac{x^2 + x \ln(1-x)}{\sin^2 x - \ln^2(1-x)} \leftarrow$$

6. $$\int \frac{\operatorname{arctg}\sqrt{x}}{(1-x)^2}\, dx$$

7. $$\sum_{n=1}^{\infty} \frac{(n^3)\sqrt[3]{n^2}}{2^n}$$

8. $$f(x,y) = x^4 + y^3 - 4x^2 - 3y^2$$

CONT. $\Rightarrow$ DERIV.

DERIV $\not\Rightarrow$ CONT.

der in 0 non cont in 0

1. $\forall \varepsilon > 0 \ \exists I(2) : \left| \log_2 (x+2) - 2 \right| < \varepsilon \quad \forall x \in I(2) - \{2\}$

$D: \quad x+2>0$

$\quad\quad x>-2$

$$\begin{cases} \log_2 (x+2) - 2 < \varepsilon \\ \log_2 (x+2) - 2 > -\varepsilon \end{cases} \qquad \begin{cases} \log_2 (x+2) < (2+\varepsilon) \cdot \log_2 2 \\ \log_2 (x+2) > (2-\varepsilon) \cdot \log_2 2 \end{cases}$$

$$\begin{cases} x+2 < 2^{2+\varepsilon} \\ x+2 > 2^{2-\varepsilon} \end{cases} \qquad \begin{cases} x < 2^{2+\varepsilon} - 2 \\ x > 2^{2-\varepsilon} - 2 \end{cases}$$

$$\underbrace{2^{2-\varepsilon} - 2} < x < \underbrace{2^{2+\varepsilon} - 2}$$

Per $\varepsilon \to 0$

(cioè ε sufficientemente piccolo gli estremi dell'intorno trovato tendono a 2, quindi abbiamo trovato un intorno di 2.)

2. $f(x) = \dfrac{2}{x\sqrt{x+4}}$

- $D: \ x \neq 0 \ \text{e} \ x > -4$

  $(-4\,;0) \cup (0, +\infty)$

- nè pari, nè dispari;

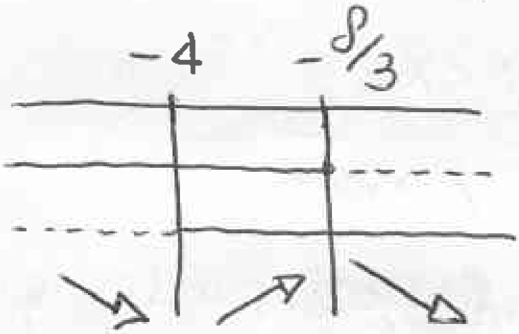
- nessun intersezione con l'asse y, perchè $x \neq 0$.

· nessun intersezione con l'asse x perchè $2 \neq 0$.

· $$f'(x) = \frac{-2\left[\sqrt{x+4} + x \cdot \frac{1}{2\sqrt{x+4}}\right]}{x^2(x+4)}$$

$$f'(x) = \frac{-\cancel{2}\left[\frac{2(x+4)+x}{\cancel{2}\sqrt{x+4}}\right]}{x^2(x+4)} = \frac{-3x-8}{x^2\sqrt{x+4}\cdot(x+4)}$$

$$f'(x) \geq 0 \iff \left\| \begin{array}{l} -3x-8 \geq 0 \\ x+4>0 \end{array} \right. \left\| \begin{array}{l} 3x+8 \leq 0 \\ x>-4 \end{array} \right. \left\| \begin{array}{l} x \leq -\frac{8}{3} \\ x>-4 \end{array} \right.$$



$$f\left(-\frac{8}{3}\right) = \frac{2}{-\frac{8}{3}\sqrt{-\frac{8}{3}+4}} = \frac{2}{-\frac{8}{3}\sqrt{\frac{-8+12}{3}}} = \frac{2}{-\frac{8}{3}\sqrt{\frac{4}{3}}}$$

$$= \frac{2}{-\frac{16}{3\sqrt{3}}} = -\cancel{2}^{1} \cdot \frac{3\sqrt{3}}{\cancel{16}\,8} = -\frac{3\sqrt{3}}{8}$$

$A\left(-\frac{8}{3}; -\frac{3\sqrt{3}}{8}\right)$ è un massimo.

$$-3x^2(x+4)\sqrt{x+4} + (3x+8)\left[\frac{4x(x+4)^2+3x^2(x+4)}{2\sqrt{x+4}}\right] \geq 0$$

$$\frac{-6x^2(x+4)^2+4x(x+4)^2(3x+8)+3x^2(x+4)(3x+8)}{2\sqrt{x+4}} \geq 0$$

$$x(x+4)\left[-6x(x+4)+4(x+4)(3x+8)+3x(3x+8)\right] \geq 0$$

$$x(x+4)\left[-6x(x+4)+(3x+8)(4x+16+3x)\right] \geq 0$$

$$x(x+4)\left[-6x(x+4)+(3x+8)(7x+16)\right] \geq 0$$

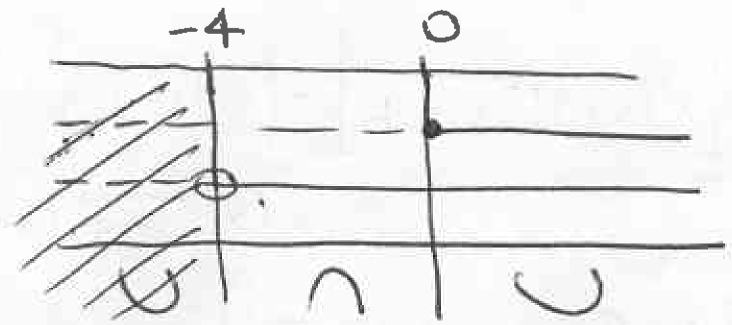
$$x(x+4)\left[-6x^2-24x+21x^2+48x+56x+128\right] \geq 0$$

$$x(x+4)(15x^2+80x+128) \geq 0$$

$$+\begin{cases} x \geq 0 \\ x \geq -4 \\ 15x^2+80x+128 \geq 0 \end{cases}$$

$\Delta = 6400 - 7680 < 0$

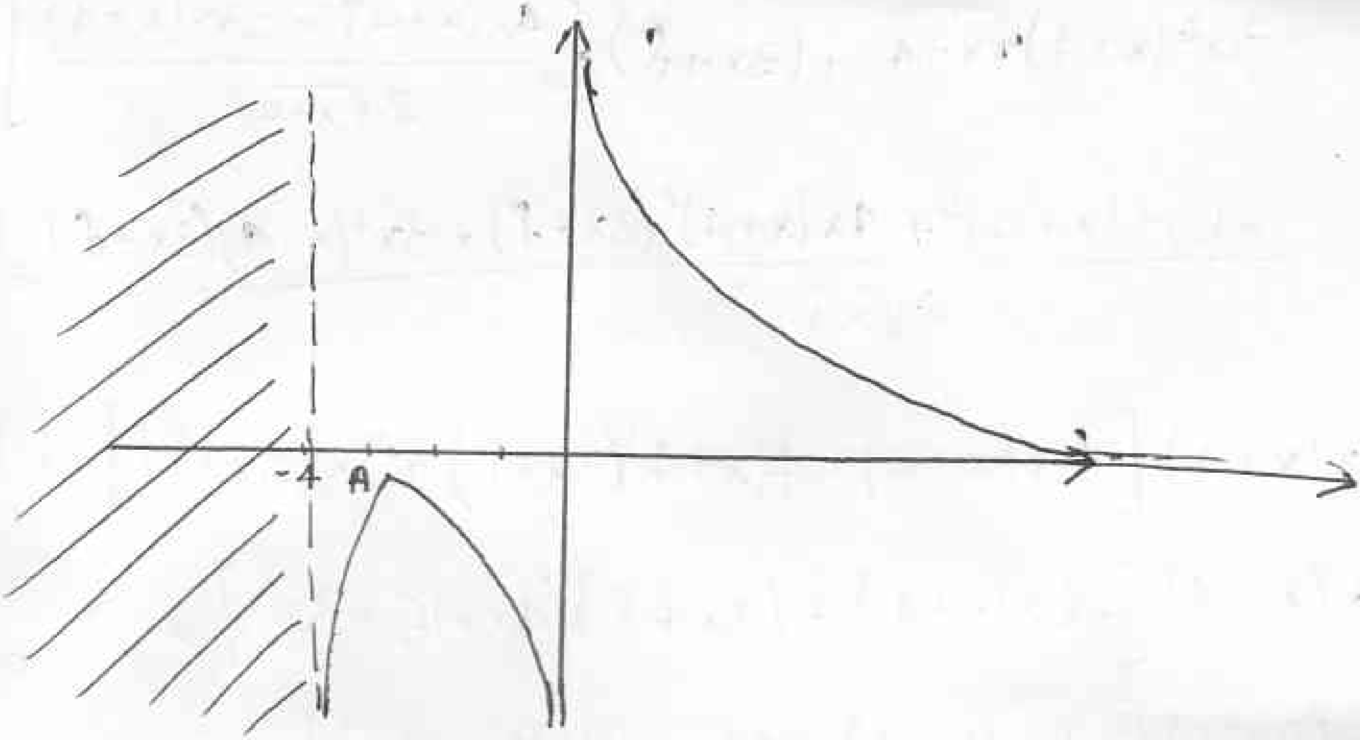
$$+\begin{cases} x \geq 0 \\ x > -4 \\ \forall x \in \mathbb{R} \end{cases}$$



Per $x<-4$ non ci interessa cosa accade.

convessa concava

• Positività di f: $f(x) > 0 \iff x > 0$



· Asintoti verticali :

$$\lim_{x\to(-4)^+} \frac{2}{x\sqrt{x+4}} = \frac{-2}{0^+} = -\infty \qquad x=-4 \text{ A.V.}$$

$$x=0 \text{ A.V.}$$

$$\lim_{x\to 0^-} \frac{2}{x\sqrt{x+4}} = \frac{1}{0^-} = -\infty$$

$$\lim_{x\to 0^+} \frac{2}{x\sqrt{x+4}} = \frac{1}{0^+} = +\infty$$

$$\cdot\ f''(x) = \frac{-3\left[x^2(x+4)\sqrt{x+4}\right] - (-3x-8)\left[2x(x+4)\sqrt{x+4} + \frac{x^2(x+4)}{2\sqrt{x+4}} + x^2\sqrt{x+4}\right]}{\left(x^2\sqrt{x+4}\cdot(x+4)\right)^2}$$

$$f''(x) \geq 0$$

$$-3x^2(x+4)\sqrt{x+4} + (3x+8)\left[\frac{4x(x+4)^2 + x^2(x+4) + 2x^2(x+4)}{2\sqrt{x+4}}\right] \geq 0$$

• Asintoti orizzontali:

$$\lim_{x \to +\infty} \frac{9}{x\sqrt{x+4}} = 0 \qquad y=0 \text{ A.O.}$$

Nessun asintoto obliquo.

3. $f'_+(0) = f'_-(0)$ $\qquad f(0) = \lim_{x \to 0^+} f(x)$

$$\cancel{f} = \cancel{\ln e} + \alpha$$

$$\alpha = 0$$

$$\left(\frac{1}{e+x}\right)_{x=0} = \left(\frac{\beta}{\cos^2 x}\right)_{x=0}$$

$\frac{1}{e} = \beta$ e ~~[illegible]~~ $\alpha = 0$

4. $\frac{3^n}{1+3^n}$ è una succ. limitata, infatti

$$0 < \frac{3^n}{1+3^n} < 1$$

Affinché sia vera la prima disequazione $\frac{3^n}{1+3^n} > 0$ entrambi i termini del rapporto devono avere lo stesso segno, il che è vero perché $3^n > 0$ e $1+3^n > 0$. Quindi $\inf A = 0$, ma non è il min ~~max~~ perché $0 \notin A$.

Inoltre si ha anche $\frac{3^n}{1+3^n} < 1$, infatti:

$$\frac{3^n-1-3^n}{1+3^n} < 0 \iff \frac{-1}{1+3^n} < 0$$, il che è vero

perché $1+3^n > 0$. Quindi $\sup A = 1$, ma $1 \notin A$

quindi $\nexists \max A$.

5. $$\lim_{x\to 0} \frac{x^2 + x\ln(1-x)}{\sin^2 x - \ln^2(1-x)} =$$

$$\lim_{x\to 0} \frac{x^2 + x\ln(1+(-x))}{\operatorname{sen}^2 x - \ln^2(1+(-x))} =$$

$$\cancel{\lim_{x\to 0} \frac{x^2 + x\left(-x - \frac{x^2}{2} + o(x^2)\right)}{\left(x - \frac{x^3}{6}\right)^2 + o(x^6) - \left(-x - \frac{x^2}{2}\right)^2 + o(x^4)}}$$

$$\lim_{x\to 0} \frac{x^2 + x\left(-x - \frac{x^2}{2} + o(x^2)\right)}{\cancel{x^2} - \frac{x^4}{3} + o(x^5) - \cancel{x^2} - x^3 + o(x^3)} =$$

$$\lim_{x\to 0} \frac{\cancel{x^2} - \cancel{x^2} - \frac{x^3}{2} + o(x^3)}{-x^3 + o(x^3)} = \lim_{x\to 0} \frac{\frac{x^3}{2} + o(x^3)}{x^3 + o(x^3)}$$

$$= \lim_{x\to 0} \frac{\cancel{x^3}\left(\frac{1}{2} + \frac{o(x^3)}{x^3}\right)}{\cancel{x^3}\left(1 + \frac{o(x^3)}{x^3}\right)} = \frac{1}{2}$$

6. $\sqrt{x} = t$

$D(\operatorname{arctg} x) = \frac{1}{1+x^2}$

$x = t^2$

$\int f'(x) g(x) = f(x) g(x) - \int f(x) g'(x)$

$dx = 2t\,dt$

$$\int \frac{\operatorname{arctg} t}{(1-t^2)^2} \cdot 2t\,dt =$$

$2t(1-t^2)^{-2}$

$D\left[+(1-t^2)^{-1}\right]$

$$\int \frac{2t}{(1-t^2)^2} \operatorname{arctg} t \, dt =$$

$$(1-t^2)^{-1} \cdot \operatorname{arctg} t - \int (1-t^2)^{-1} \cdot \frac{1}{1+t^2} dt =$$

$$\frac{\operatorname{arctg} t}{1-t^2} - \int \frac{1}{(1-t^2)(1+t^2)}^{*} dt =$$

$$\frac{\operatorname{arctg} t}{1-t^2} + \frac{1}{4} \log \left| \frac{1-t}{1+t} \right| - \frac{1}{2} \operatorname{arctg} t + C, C \in \mathbb{R}$$

$$(\operatorname{arctg} t)\left[\frac{1}{1-t^2} - \frac{1}{2}\right] + \frac{1}{4} \log \left| \frac{1-t}{1+t} \right| + C, C \in \mathbb{R}$$

$$(\operatorname{arctg} t)\left[\frac{2-1+t^2}{2(1-t^2)}\right] + \frac{1}{4} \log \left| \frac{1-t}{1+t} \right| + C, C \in \mathbb{R}$$

$$= (\operatorname{arctg} t)\left(\frac{1+t^2}{2(1-t^2)}\right) + \frac{1}{4}\log\left|\frac{1-t}{1+t}\right| + c, \; c \in \mathbb{R}$$

$t = \sqrt{x}$

$$= (\operatorname{arctg} \sqrt{x})\left(\frac{1+x}{2(1-x)}\right) + \frac{1}{4}\log\left|\frac{1-\sqrt{x}}{1+\sqrt{x}}\right| + c, \; c \in \mathbb{R}.$$

$$* \quad \frac{1}{(1-t^2)(1+t^2)} = \frac{1}{(1-t)(1+t)(1+t^2)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C(2t)+D}{1+t^2}$$

$$= \frac{A(1+t)(1+t^2) + B(1-t)(1+t^2) + (2Ct+D)(1-t^2)}{1-t^4} =$$

$$= \frac{A(1+t^2+t+t^3) + B(1+t^2-t-t^3) + (2Ct-2Ct^3+D-Dt^2)}{1-t^4}$$

$$= \frac{A+At^2+At+At^3+B+Bt^2-Bt-Bt^3+2Ct-2Ct^3+D-Dt^2}{1-t^4}$$

$$= \frac{(A-B-2C)t^3+(A+B-D)t^2+(A-B+2C)t+(A+B+D)}{1-t^4}$$

$$\begin{cases} A-B-2C=0 \\ A+B-D=0 \\ A-B+2C=0 \\ A+B+D=1 \end{cases} \quad \begin{cases} C=0 \\ A=B \\ A+B-D=0 \\ A+B+D=1 \end{cases} \quad \begin{cases} C=0 \\ A=B \\ -2D=-1 \\ A+B=D \end{cases}$$

$$\begin{cases} C=0 \\ A=B \\ D=1/2 \\ 2A=1/2 \end{cases} \quad \begin{cases} C=0 \\ B=\frac{1}{4} \\ D=\frac{1}{2} \\ A=\frac{1}{4} \end{cases}$$

$$\int \frac{1}{1-t^4}\,dt = \frac{1}{4}\int \frac{1}{1-t}\,dt + \frac{1}{4}\int \frac{1}{1+t}\,dt + \frac{1}{2}\int \frac{1}{1+t^2}\,dt$$

$$= -\frac{1}{4}\log|1-t| + \frac{1}{4}\log|1+t| + \frac{1}{2}\operatorname{arctg} t + C, \quad C\in\mathbb{R}$$

7. $$\sum_{n=1}^{+\infty} \frac{n^3 \sqrt[3]{n^2}}{2^n} = \sum_{n=1}^{+\infty} \frac{\sqrt[3]{n^4}}{2^n}$$

$a_n > 0$

Criterio del rapporto:

$$\lim_{n \to +\infty} \frac{\sqrt[3]{(n+1)^4}}{2^{n+1}} \cdot \frac{2^n}{\sqrt[3]{n^4}} = \frac{1}{2} \lim_{n \to +\infty} \left(\frac{n+1}{n}\right)^{\frac{4}{3}}$$

$= \frac{1}{2} < 1 \Rightarrow$ La serie conv.

8. $$f(x,y) = x^4 + y^3 - 4x^2 - 3y^2$$

$D: \mathbb{R}^2$

$f_x(x,y) = 4x^3 - 8x$ $\quad f_y(x,y) = 3y^2 - 6y$

$f_{xx}(x,y) = 12x^2 - 8$ $\quad f_{yy}(x,y) = 6y - 6$

$f_{xy} = 0$ $\quad f_{yx} = 0$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \quad \begin{cases} 4x^3 - 8x = 0 \\ 3y^2 - 6y = 0 \end{cases} \quad \begin{cases} 4x(x^2 - 2) = 0 \\ 3y(y-2) = 0 \end{cases}$$

$$\begin{cases} x = 0,\ x = \sqrt{2},\ x = -\sqrt{2} \\ y = 0,\ y = 2 \end{cases}$$

$A(0,0)$, $B(0,2)$, $C(\sqrt{2},0)$

$D(\sqrt{2},2)$, $E(-\sqrt{2},0)$, $F(-\sqrt{2},2)$

$$Hf(x,y) = \begin{vmatrix} 12x^2-8 & 0 \\ 0 & 6y-6 \end{vmatrix} = (12x^2-8)(6y-6)$$

$Hf(0,0) = 48 > 0 \quad f_{xx}(0,0) = -8 < 0 \Rightarrow A(0,0)$ max relativo

$Hf(0,2) = (-8)(12-6) = (-8)\cdot 6 = -48 < 0 \Rightarrow B(0,2)$ nè max, nè min

$Hf(\sqrt{2},0) = 16(-6) < 0 \Rightarrow C(\sqrt{2},0)$ nè max, nè min.

$Hf(\sqrt{2},2) = 16\cdot 6 > 0 \quad f_{xx}(\sqrt{2},2) = 16 > 0 \Rightarrow D(\sqrt{2},2)$ min relativo

$Hf(-\sqrt{2},0) = 16\cdot(-6) < 0 \Rightarrow E(-\sqrt{2},0)$ nè max, nè min.

$Hf(-\sqrt{2},2) = 16\cdot 6 > 0 \quad f_{xx}(-\sqrt{2},2) = 16 > 0 \Rightarrow F(-\sqrt{2},2)$ min rel.

$f(\sqrt{2},2) = \sqrt{2}^{4} + \cancel{8} - \cancel{8} - 12 = 4 - 12 = -8$

$f(-\sqrt{2},2) = (-\sqrt{2})^4 + \cancel{8} - \cancel{8} - 12 = -8$

$A(0,0)$ max assoluto.

$D(\sqrt{2},2)$ min relativo.

$F(-\sqrt{2},2)$ min relativo.